Representanions of Hardy algebras associated with W^*-correspondences

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Outline

1. The Tensor and Hardy Algebras
2. Representations of the Tensor algebras
3. Representations of the Hardy algebras
4. Noncommutative Nevanlinna-Pick theorem

The Setup

We begin with the following setup:

- M - a W^*-algebra.
- E - a W^*-correspondence over M. This means that E is a **bimodule** over M which is endowed with an M-**valued inner product** (making it a right-Hilbert C^*-module that is self dual). The left action of M on E is given by a unital, normal, *-homomorphism φ of M into the (W^*-) algebra of all bounded adjointable operators $\mathcal{L}(E)$ on E.

Examples

- (Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$, $d \geq 1$.
- $G = (G^0, G^1, r, s)$- a finite directed graph. $M = \ell^\infty(G^0)$, $E = \ell^\infty(G^1)$, $a\xi b(e) = a(r(e))\xi(e)b(s(e))$, $a, b \in M$,$\xi \in E$ $\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \langle \xi(e), \eta(e) \rangle$, $\xi, \eta \in E$.
- M- arbitrary , $\alpha : M \to M$ a normal endomorphism. $E = M$ with right action by multiplication, left action by $\varphi = \alpha$ and inner product $\langle \xi, \eta \rangle := \xi^* \eta$. Denote it ${}_\alpha M$.
- Φ is a normal, contractive, CP map on M. $E = M \otimes_\Phi M$ is the completion of $M \otimes M$ with $\langle a \otimes b, c \otimes d \rangle = b^* \Phi(a^* c) d$ and $c(a \otimes b)d = ca \otimes bd$.

Note: If σ is a representation of M on H, $E \otimes_\sigma H$ is a Hilbert space with $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle_E) h_2 \rangle_H$.

Similarly, given two correspondences E and F over M, we can form the (internal) tensor product $E \otimes F$:

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle = \langle f_1, \varphi(\langle e_1, e_2 \rangle_E) f_2 \rangle_F$$

$$\varphi_{E \otimes F}(a)(e \otimes f)b = \varphi_E(a)e \otimes fb.$$

In particular we get "tensor powers" $E^{\otimes k}$.
Also, given a sequence $\{E_k\}$ of correspondences over M, the direct sum $E_1 \oplus E_2 \oplus E_3 \oplus \cdots$ is also a correspondence (after an appropriate completion).

For a correspondence E over M we define the Fock correspondence

$$\mathcal{F}(E) := M \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots$$

For every $a \in M$ define the operator $\varphi_\infty(a)$ on $\mathcal{F}(E)$ by

$$\varphi_\infty(a)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n$$

and $\varphi_\infty(a)b = ab$.
For $\xi \in E$, define the “shift” (or “creation”) operator T_ξ by

$$T_\xi(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n.$$

and $T_\xi b = \xi b$. So that T_ξ maps $E^{\otimes k}$ into $E^{\otimes (k+1)}$.

Definition

(1) The norm-closed algebra generated by $\varphi_\infty(M)$ and $\{T_\xi : \xi \in E\}$ will be called the **tensor algebra** of E and denoted $\mathcal{T}_+(E)$.

(2) The ultra-weak closure of $\mathcal{T}_+(E)$ will be called the **Hardy algebra** of E and denoted $H^\infty(E)$.

Examples

1. If $M = E = \mathbb{C}$, $\mathcal{F}(E) = \ell^2$, $\mathcal{T}_+(E) = A(\mathbb{D})$ and $H^\infty(E) = H^\infty(\mathbb{D})$.
2. If $M = \mathbb{C}$ and $E = \mathbb{C}^d$ then $\mathcal{F}(E) = \ell^2(\mathbb{F}_d^+)$, $\mathcal{T}_+(E)$ is Popescu's $\mathcal{A}_d$ and $H^\infty(E)$ is F_d^∞ (Popescu) or $\mathcal{L}_d$ (Davidson-Pitts).

Representations

In the purely algebraic setting the representations of the Tensor algebra are given by bimodule maps. Here

Definition

Let E be a W^*-correspondence over a von Neumann algebra M. Then:
completely contractive covariant representation of E on a Hilbert space H is a pair (T,σ), where

1. σ is a normal $*$-representation of M in $B(H)$.
2. T is a linear, completely contractive map from E to $B(H)$ that is continuous in the σ-topology on E and the ultraweak topology on $B(H)$.
3. T is a bimodule map in the sense that $T(S\xi R) = \sigma(S)T(\xi)\sigma(R)$, $\xi \in E$, and $S, R \in M$.

Theorem

Given a c.c. representation (T,σ) of E on H, there is a unique c.c. representation of $\mathcal{T}_+(E)$ on H, written $T\times\sigma$, such that

$$T\times\sigma(\varphi_\infty(a)) = \sigma(a), \quad a\in M$$

$$T\times\sigma(T_\xi) = T(\xi), \quad \xi\in E.$$

Conversely, every c.c. representation of $\mathcal{T}_+(E)$ is of this form.

Note: Given a c.c. representation (T,σ) on H, one can define a contraction

$$\tilde{T} : E\otimes_\sigma H \to H$$

by $\tilde{T}(\xi\otimes h) = T(\xi)h$ and $\tilde{T}$ has the intertwining property

$$\tilde{T}(\varphi(a)\otimes I_H) = \sigma(a)\tilde{T}, \quad a\in M.$$

Conversely: Every such map defines a representation of E.

The σ-dual E^σ

Recall: $\tilde{T} : E \otimes_\sigma H \to H$ and $\tilde{T}(\varphi(a) \otimes I_H) = \sigma(a)\tilde{T}$, $a \in M$.

Corollary

The c.c. representations of $\mathcal{T}_+(E)$ that restrict to σ on M are parameterized by maps $\eta : H \to E \otimes_\sigma H$ with $\|\eta\| \leq 1$ that satisfy

$$(\varphi(a) \otimes I_H)\eta = \eta\sigma(a), \quad a \in M.$$

Note: $\eta = \tilde{T}^*$. The reason to look at the adjoint of these maps is:

Lemma

The set of all the maps $\eta : H \to E \otimes_\sigma H$ with the above intertwining property is a W^*-correspondence over $\sigma(M)'$ with inner product $\langle \eta, \zeta \rangle = \eta^*\zeta$ and bimodule structure $b \cdot \eta c = (I \otimes b)\eta c$ $(b, c \in \sigma(M)')$. Write E^σ for it and call it **the σ-dual of** E.

Corollary

The closed unit ball, $\overline{\mathbb{D}(E^\sigma)}$ parameterizes the c.c. representations of $\mathcal{T}_+(E)$.

Examples

(1) $M = E = \mathbb{C}$. So $\mathcal{T}_+(E) = A(\mathbb{D})$, σ is on H and $E^\sigma = B(H)$.

(2) $M = \mathbb{C}$, $E = \mathbb{C}^d$. $\mathcal{T}_+(E) = \mathcal{A}_d$ (Popescu's algebra) and $E^\sigma = (B(H))^{(d)} : H \to \mathbb{C}^d \otimes H$. c.c. representations are parameterized by row contractions $(T_1, \ldots, T_d)$.

(3) M general, $E =_\alpha M$ for an automorphism α. $\mathcal{T}_+(E) =$ the analytic crossed product.
$E^\sigma = \{X \in B(H) : \ \sigma(\alpha(T))X = X\sigma(T), T \in B(H)\}$.

Ultraweakly continuous representations of $H^\infty(E)$

Question: For which $\eta \in \overline{\mathbb{D}(E^\sigma)}$ does the corresponding representation extend to an ultra-weakly continuous, c.c. representation of $H^\infty(E)$?

Recall :

When $M = E = \mathbb{C}$, $H^\infty(E) = H^\infty(\mathbb{D})$. If σ is on H, $E^\sigma = B(H)$ so $\eta \in B(H)$ and $\|\eta\| \leq 1$. Then η defines a c.c. representation of $A(\mathbb{D})$ and the question is: When does it have an H^∞-functional calculus?

The answer: if and only if the spectral measure of the unitary part of the operator is absolutely continuous with respect to Lebesgue measure.

A completely contractive covariant representation (T, σ) of (E, M) on a Hilbert space H is called **isometric** in case $\widetilde{T} : E \otimes H \to H$ is an isometry.

Theorem

Let (T, σ) be a completely contractive covariant representation of (E, M) on a Hilbert space H. Then there is an isometric covariant representation (V, ρ) of (E, M) acting on a Hilbert space K containing H such that if P denotes the projection of K onto H, then

1. *P commutes with $\rho(M)$ and $\rho(a)P = \sigma(a)P$, $a \in M$, and*
2. *for all $\eta \in E$, $V(\eta)^*$ leaves H invariant and $PV(\eta)P = T(\eta)P$.*

The representation (V, ρ) may be chosen so that the smallest subspace of K that contains H is all of K. When this is done, (V, ρ) is unique up to unitary equivalence and is called **the minimal isometric dilation** *of (T, ρ).*

Remarks :

(i) If (V,ρ) dilates (T,σ) as above, $P(V\times\rho)P = T\times\sigma$. Thus, if (V,ρ) extends, so does (T,σ).

(ii) **Induced representations**: Fix a faithful representation π of M on K, let $H=\mathcal{F}(E)\otimes_\pi K$ and define the representation of E by $\sigma(a)=\varphi_\infty(a)\otimes I_K$, $T(\xi)=T_\xi\otimes I_K$. This extends to an ultra weakly continuous representation of $H^\infty(E)$ by $X\mapsto X\otimes I_K$.

(iii) If $\|\tilde{T}\|=\|\eta\|<1$, the representation can be dilated to an induced representation. Thus extends.

The problem is with η on the boundary.

Following Davidson-Li-Pitts we define

Definition

Given a c.c. representation (T, σ) on H, we say that $x \in H$ is **absolutely continuous** if the functional $\omega_x \circ (T \times \sigma)$, on $\mathcal{T}_+(E)$, extends to a ultraweakly continuous functional on $H^\infty(E)$ and we write $\mathcal{V}_{ac}(T, \sigma)$ for the set of all the absolutely continuous vectors for (T, σ).

Here $\omega_x(Y) = \langle Yx, x \rangle$.
Notation : Fix a faithful normal representation of infinite multiplicity π of M on K_0 and write (S_0, σ_0) the associated induced representation (on $H_0 = \mathcal{F}(E) \otimes_\pi K_0$). For a given $\eta \in \overline{\mathbb{D}(E^\sigma)}$ (corresponding to the representation (T, σ) on H) write $\mathcal{I}(S_0, \eta)$ for the space of intertwiners:

$$\{C : H_0 \to H : CS_0(\xi) = T(\xi)C,\ C\sigma_0(a) = \sigma(a)C, \xi \in E,\ a \in M\}.$$

Theorem

If (T, σ) is an isometric representation then

$$\mathcal{V}_{ac}(T,\sigma) = \bigcup \{Ran(C) : C \in \mathcal{I}(S_0, \tilde{T})\}.$$

It follows that $\mathcal{V}_{ac}(T,\sigma)$ is a closed, $\sigma(M)$-invariant subspace.

Sketch of the proof of $\subseteq$: Fix $x \in \mathcal{V}_{ac}(T,\sigma)$.

(i) Can find $g, k \in \mathcal{F}(E) \otimes_\pi K_0$ such that, for $X \in \mathcal{T}_+(E)$,

$$\langle (T \times \sigma)(X)x, x\rangle = \langle (S_0 \times \sigma_0)(X)g, k\rangle$$

and then, for $\tau := (T \times \sigma) \oplus (S_0 \times \sigma_0)$,

$$\langle \tau(X)(x \oplus g), (x \oplus (-k))\rangle = 0.$$

(ii) Write $\mathcal{N} = [\tau(\mathcal{T}_+(E))(x \oplus g)]$ and (R, ρ) for $\tau|\mathcal{N}$.

(iii) Using the Wold decomposition and part (i),we show that (R, ρ) is induced.

(iv) $C_0 := P_H|\mathcal{N}$ intertwines (R, ρ) and (T, σ) and can be extended to $C \in \mathcal{I}(S_0, \tilde{T})$. Also $C_0(x \oplus g) = x$.

General c.c. representations

Theorem

Let (T, σ) be a completely contractive representation of (E, M) on the Hilbert space H, let (V, ρ) be the minimal isometric dilation of (T, σ) acting on a Hilbert space K containing H, and let P denote the projection of K onto H. Then $K \ominus H$ is contained in $\mathcal{V}_{ac}(V, \rho)$ and the following sets are equal.

(1) $\mathcal{V}_{ac}(T, \sigma)$.

(2) $H \cap \mathcal{V}_{ac}(V, \rho)$.

(3) $P\mathcal{V}_{ac}(V, \rho)$.

(4) $\bigcup\{Ran(C) \mid C \in \mathcal{I}(S_0, \tilde{T})\}$.

In particular, $\mathcal{V}_{ac}(T, \sigma) = H$ if and only if $\mathcal{V}_{ac}(V, \rho) = K$.

Intertwiners and superharmonic elements

Given a c.c. representation (T, σ), one defines the completely positive map associated with it, $\Phi_T : \sigma(M)' \to \sigma(M)'$ by

$$\Phi_T(b) = \eta^*(I_E \otimes b)\eta = \tilde{T}(I_E \otimes b)\tilde{T}^*$$

where $\eta = \tilde{T}^*$.
E.g.: If $\tilde{T} = (T_1, \ldots, T_d)$ (a row-contraction),

$$\Phi_T(b) = \sum T_i b T_i^*.$$

Note: If $C \in \mathcal{I}(S_0, \eta)$, then $Q = CC^*$ lies in $\sigma(M)'$ and satisfies

* $Q \geq 0$ and $\Phi_T(Q) \leq Q$, and
* $\Phi_T^n(Q) \to 0$ ultra weakly.

Such an element will be said to be *pure superharmonic* for Φ_T.

The converse also holds: Given an element $Q \in \sigma(M)'$ that is pure superharmonic for Φ_T, define $r \in \sigma(M)'$ to be the positive square root of $Q - \Phi_T(Q)$. Then $\sum_{n \geq 0} \Phi_T^n(r^2) = Q$ (SOT). Let $\mathcal{R}$ be the closure of the range of r and $\sigma_{\mathcal{R}} := \sigma|\mathcal{R}$. Let $v : \mathcal{R} \to K_0$ be an isometry that intertwines $\sigma_{\mathcal{R}}$ and π. Then set

$$C^* = (I_{\mathcal{F}(E)} \otimes v) \sum_{n \geq 0} (I_{E^{\otimes n}} \otimes r) \tilde{T}_n^*$$

where $\tilde{T}_n = \tilde{T}(I_E \otimes \tilde{T}) \cdots (I_{E^{\otimes(n-1)}} \otimes \tilde{T}) : E^{\otimes n} \otimes_\sigma H \to H$. We have $C \in \mathcal{I}(S_0, \tilde{T})$ and $CC^* = Q$.

Corollary

$$\bigcup \{Ran(C) : C \in \mathcal{I}(S_0, \tilde{T})\} =$$

$$= \bigvee \{Ran(Q) : Q \textit{ is a pure superharmonic operator for } \Phi_T\}.$$

Theorem

Let $T \times \sigma$ be a c.c. representation of $\mathcal{T}_+(E)$ on H and write $\eta = \tilde{T}^*$ for the element of $\overline{\mathbb{D}(E^\sigma)}$ associated with it. Then the following are equivalent.

- *The representation $T \times \sigma$ extends to a c.c. ultra weakly continuous representation of $H^\infty(E)$.*
- $\mathcal{V}_{ac}(T, \sigma) = H$
- $H = \bigvee\{Ran(C) : C \in \mathcal{I}(S_0, \eta)\}$.
- $H = \bigvee\{Ran(Q) : Q \text{ is pure superharmonic for } \Phi_T\}$

Arguments and partial results: Douglas (69), Davidson-Li-Pitts (05).
This theorem describes the representations of $H^\infty(E)$, whose restriction to M is given (equals σ), as a subset $AC(E^\sigma)$ of the ball $\overline{\mathbb{D}(E^\sigma)}$ containing $\mathbb{D}(E^\sigma)$. Recall that E^σ is a W^*-correspondence.

Elements of $H^\infty(E)$ as functions

Conclusion: We now view the elements of $H^\infty(E)$ as functions ($B(H)$-valued)on $AC(E^\sigma)$. For $X \in H^\infty(E)$, we write $\hat{X}$ for the resulting function. Thus

$$\widehat{X}(\eta^*) = (\eta^* \times \sigma)(X).$$

Next, I will present an interpolation result (the noncommutative Nevanlinna-Pick theorem) for these functions (**on** $AC(E^\sigma)$). For the open unit ball, this was done in the case $M = \mathbb{C}$ (and others) by Popescu and by Davidson-Pitts. The general situation (on the open ball) was proved by us in 2004.

Noncommutative Nevanlinna-Pick interpolation

The classical NP theorem Given $z_1, \ldots, z_m$ in $\mathbb{D} \subseteq \mathbb{C}$ and $w_1, \ldots, w_m$ in $\mathbb{C}$, when can one find a function $f \in H^\infty(\mathbb{D})$ with $\|f\| \leq 1$ and $f(z_i) = w_i$ for all i?
Answer: If and only if the $m \times m$ matrix (the Pick matrix)

$$\left(\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}}\right)$$

is positive.

Our noncommutative analogue on the **open** unit ball is the following.

Theorem

Let E be a W^*-correspondence over M, σ a faithful normal representation of M on H. For $\eta_1, \dots, \eta_m \in \mathbb{D}(E^\sigma)$ and $D_1, \dots, D_m \in B(H)$, one can find $X \in H^\infty(E)$ with $\|X\| \le 1$ that satisfies

$$\widehat{X}(\eta_i^*) = D_i$$

for all $i \le m$ if and only if the $m \times m$ matrix of maps

$$((Id - Ad(D_i, D_j)) \circ (Id - \theta_{\eta_i,\eta_j})^{-1})$$

defines a completely positive map from $M_m(\sigma(M)')$ to $M_m(B(H))$.

For $M = \mathbb{C}, E = \mathbb{C}^n$: Popescu, Davidson-Pitts. **Question :**What about $\eta_1, \dots, \eta_m$ in $AC(\sigma)$?
In this case $(Id - \theta_{\eta_i,\eta_j})^{-1}$ is not defined.

Simple observation : If Φ, Ψ are positive maps such that $Id - \Phi$ is invertible then $(Id - \Psi) \circ (Id - \Phi)^{-1}$ is positive if and only if :

$$\{a \geq 0 : \Phi(a) \leq a\} \subseteq \{a \geq 0 : \Psi(a) \leq a\}.$$

The last statement makes sense even if $Id - \Phi$ is not invertible. It is related to the Lyapunov preorder studied in matrix theory. This was pointed out to us by Nir Cohen.

Theorem

Let E be a W^*-correspondence over the von Neumann algebra M and let σ be a faithful normal representation of M on H. Given $\eta_1, \eta_2, \dots, \eta_k \in AC(E^\sigma)$ and $D_1, D_2, \dots, D_k \in B(H)$, the following conditions are equivalent.

(1) *There is an element $X \in H^\infty(E)$ such that $\|X\| \leq 1$ and such that*

$$X(\eta_i^*) = D_i \ ,$$

$i = 1, 2, \dots, k$.

(2) *For every $m \geq 1$, $i : \{1, \dots, m\} \to \{1, \dots, k\}$ and $C_1, C_2, \dots, C_m$ with $C_j \in \mathcal{I}(S_0, \eta_{i(j)}^*)$, we have*

$$(D_{i(l)} C_l C_j^* D_{i(j)}^*)_{l,j} \leq (C_l C_j^*)_{l,j}.$$

About the proof of the NP Theorem : The proof follows the general lines of Sarason's proof (1967) of the classical NP Theorem. The main two ingredients used are:

(1) The Commutant Lifting Theorem for c.c. representations: Every operator that commutes with the image of the representation can be lifted to an operator that commutes with the image of the minimal isometric dilation without increasing the norm.

(2) Identification of the commutant.

The commutant of an induced representation

Let E be a W^*-correspondence over the von Neumann algebra M and let σ be a faithful normal representation of M on H. The algebra $H^\infty(E)$ can be represented (completely isometrically, ultra-weakly homeomorphically) on $\mathcal{F}(E) \otimes_\sigma H$ by $X \mapsto X \otimes I_H$.
Write $H^\infty(E) \otimes I_H$ for the image of this representation.
Similarly, we can write $H^\infty(E^\sigma) \otimes I_H$ for the algebra $H^\infty(E^\sigma)$ represented on $\mathcal{F}(E^\sigma) \otimes_\iota H$ (where ι is the identity representation of $\sigma(M)'$ on H).

Theorem

The commutant of $H^\infty(E) \otimes I_H$ is unitarily isomorphic to $H^\infty(E^\sigma) \otimes I_H$.
Consequently (by duality), $(H^\infty(E) \otimes I_H)'' = H^\infty(E) \otimes I_H$.

Thank You !